Face numbers of sequentially Cohen-Macaulay complexes and Betti numbers of componentwise linear ideals

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Abstract

A numerical characterization is given of the so-called h-triangles of sequentially Cohen-Macaulay simplicial complexes. This result characterizes the number of faces of various dimensions and codimensions in such a complex, generalizing the classical Macaulay-Stanley theorem to the nonpure case. Moreover, we characterize the possible Betti tables of componentwise linear ideals. A key tool in our investigation is a bijection between shifted multicomplexes of degree $\leq d$ and shifted pure $(d-1)$-dimensional simplicial complexes.

1 Introduction

The notion of sequentially Cohen–Macaulay complexes first arose in combinatorics: Motivated by questions concerning subspace arrangements, Björner & Wachs introduced the notion of nonpure shellability [BW96, BW97]. Stanley then introduced the sequentially Cohen–Macaulay property in order to have a ring-theoretic analogue of nonpure shellability [Sta96]. Schenzel independently defined the notion of sequentially Cohen–Macaulay modules (called by him Cohen–Macaulay filtered modules [Sch99]), inspired by earlier work of Goto. In essence, a simplicial complex is sequentially Cohen–Macaulay if and only if it is naturally composed of a sequence of Cohen–Macaulay subcomplexes, namely the pure skeleta of the complex, graded by dimension. They come with an associated numerical invariant, the so-called h-triangle, which measures the face-numbers of each component according to a doubly-indexed grading. Just as the classical h-vector determines the numbers of faces of various dimensions of a simplicial complex, the h-triangle determines the numbers of faces in each component of the complex.

Motivated by the Macaulay-Stanley theorem for Cohen-Macaulay complexes, which are always pure) Björner & Wachs [BW96] posed the problem to characterize the possible h-triangles of sequentially Cohen–Macaulay simplicial complexes. Via a connection that seems to have up to now been overlooked, this problem is equivalent to characterizing the possible Betti tables of componentwise linear ideals, see for instance [CHH04, Theorem 2.3], [KK12] and [HRW99, Proposition 12]. After some significant initial progress, due to Duval [Duv96] and Aravoma, Herzog & Hibi [AHH00], which reduced these two questions to combinatorial settings, some partial results on the second question were obtained by Crupi & Utano [CU03] and Herzog, Sharifan & Varbaro [HSV14]. Part of the difficulty of this nonpure "Macaulay problem" is that, in contrast to the classical situation, it does not suffice to use a criterion that makes a decision by only pairwise "Macaulay type" comparisons of entries in the h-triangle.

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Our main objective in this paper is to give a numerical characterization of the possible h-triangles of sequentially Cohen–Macaulay complexes. The method that we use is based on a modification of a correspondence between shifted multicomplexes and pure shifted simplicial complexes, provided by Björner, Frankl & Stanley [BFS87]. Finally, we also provide a characteristic-independent characterization of the possible Betti tables of componentwise linear ideals using our main result and an observation made by Herzog, Sharifan & Varbaro [HSV14].

The paper is organized as follows. In Section 2, we present some basic definitions and derive some necessary relations on the face numbers of sequentially Cohen–Macaulay complexes. Section 3 is devoted to our study of the Björner, Frankl & Stanley (BFS) bijection, which we examine via a new connection to lattice paths. The numerical characterization of the possible h-triangles of sequentially Cohen–Macaulay complexes is the subject of Section 4. Finally, in Section 5 we present a numerical characterization of the possible Betti tables of componentwise linear ideals.

2   Preliminaries

Simplicial complexes.   A family Δ of subsets of the set $[n] := \{1, 2, \ldots, n\}$ is called a *simplicial complex* on $[n]$, if Δ is closed under taking subsets, i.e. if $F \in \Delta$ and $F' \subseteq F$, then $F' \in \Delta$. The members F of Δ are called *faces* of Δ. The *facets* of Δ are the inclusion-wise maximal faces; the set of all facets of Δ is denoted by $\mathcal{F}(\Delta)$. The *dimension* $\dim F$ of a face F is one less than its cardinality and the *dimension* of Δ is defined to be the maximal dimension of a face. A simplicial complex of dimension $d - 1$ will be called a *$(d-1)$-complex*. A $(d-1)$-complex Δ is called *pure*, if each facet of Δ has dimension $d - 1$.

For a $(d-1)$-complex Δ, let $\Delta^i := \{\ i\text{-dimensional faces of } \Delta\ \}$ and let $f_i := |\Delta^i|$. The vector $\mathrm{f}(\Delta) = (f_{-1}, f_0, \ldots, f_{d-1})$ is called the *f-vector* of Δ. The subcomplex $\Delta^{(i)} := \bigcup_{j \leq i} \Delta^j$ is called the *i-skeleton* of Δ. The *pure i-skeleton* $\Delta^{[i]}$ of Δ is the pure i-complex whose set of facets is the set of i-dimensional faces of Δ, that is $\mathcal{F}(\Delta^{[i]}) = \Delta^i$. The *$h$-vector* $\mathrm{h}(\Delta) = (h_0, h_1, \ldots, h_d)$ of Δ is defined by

$$\sum_{i=0}^{d} h_i y^i \ = \ \sum_{i=0}^{d} f_{i-1}(1-y)^{d-i} y^i.$$

For a $(d-1)$-complex Δ, let $\widetilde{h}_{i,j} = \widetilde{h}_{i,j}(\Delta) = h_j(\Delta^{[i-1]})$. Then the triangular integer array $\widetilde{\mathrm{h}}(\Delta) = (\widetilde{h}_{i,j})_{0 \leq j \leq i \leq d}$ is called the *h-triangle* of Δ. Also, define $h_{i,j}$ by the relation

$$h_{i,j} = \widetilde{h}_{i,j} - \sum_{\ell=0}^{j} \widetilde{h}_{i+1,\ell}. \tag{1}$$

The triangular integer array $\mathrm{h}(\Delta) = (h_{i,j})_{0 \leq j \leq i \leq d}$ is called the *h-triangle* of Δ. Note that our definition of the h-triangle is equivalent to the one presented in [BW96, Definition 3.1].

Let $\Bbbk$ be an infinite field and $S = \Bbbk[x_1, x_2, \ldots, x_n]$ the polynomial ring over n variables. For a simplicial complex Δ on $[n]$, let I_Δ be the *Stanley–Reisner ideal* of Δ, that is the ideal

$$I_\Delta := \langle \{x_{i_1} x_{i_2} \ldots x_{i_r} \ : \ \{i_1, i_2, \ldots, i_r\} \notin \Delta\} \rangle$$

of S. The quotient ring $\Bbbk[\Delta] := S/I_\Delta$ is called the *face ring* of Δ. The complex Δ is said to be *Cohen–Macaulay over* $\Bbbk$, if $\Bbbk[\Delta]$ is Cohen–Macaulay (see e.g. [HH11, page 273], for Cohen–Macaulay rings). A topological characterization of the Cohen–Macaulay complexes can be found in the book by Stanley [Sta96]. The reference to the base field will usually be dropped and we simply say that Δ is Cohen–Macaulay, or CM for short.

A $(d-1)$-complex Δ is said to be *sequentially Cohen–Macaulay*, or SCM for short, if the pure i-skeleton $\Delta^{[i]}$ of Δ is CM for all $i \leq d - 1$.

A simplicial complex Δ on $[n]$ is called *shifted* if for all integers r and s with $1 \leq r < s \leq n$ and all faces F of Δ such that $r \in F$ and $s \notin F$ one has $(F \setminus \{r\}) \cup \{s\} \in \Delta$. Recall that every shifted complex is (non-pure) shellable [BW97], and a shifted complex is CM if and only if it is pure.

Face numbers of CM complexes. Let $W_n = \{w_1, w_2, \ldots, w_n\}$ be a set of variables. A *multicomplex* M on $V \subseteq W_n$ is a collection of monomials on V that is closed under divisibility. A multicomplex M on V is said to be *shifted*, if for all x_r and x_s in V with $r < s$ and all monomials m in M divisible by x_r one has that $x_s \cdot (\mathrm{m}/x_r) \in \mathrm{M}$.

Let M^i denote the set of monomials in M of degree i. The sequence $\mathrm{f}(\mathrm{M}) = (f_0, f_1, \ldots)$ is called the *f-vector* of M, where $f_i = |\mathrm{M}^i|$. The numerical characterization of f-vectors of multicomplexes (due to Macaulay [Mac27]) can be seen as the historical starting point for a line of research that this investigation is part of.

The *ℓ-representation* of a positive integer p is the unique way of writing

$$p \;=\; \binom{a_\ell}{\ell} + \binom{a_{\ell-1}}{\ell-1} + \ldots + \binom{a_e}{e},$$

where $a_\ell > a_{\ell-1} > \ldots > a_e \geq e \geq 1$. Define

$$\partial^\ell(p) \;=\; \binom{a_\ell - 1}{\ell - 1} + \binom{a_{\ell-1} - 1}{\ell - 2} + \ldots + \binom{a_e - 1}{e - 1}.$$

Also set $\partial^\ell(0) = 0$ for all ℓ. A vector $\mathrm{f} = (f_0, f_1, \ldots)$ of non-negative integers is called an *M-sequence* if $f_0 = 1$ and $\partial^\ell(f_\ell) \leq f_{\ell-1}$ for all ℓ.

A complete characterization of the h-vectors of CM complexes is achieved by combining the results by Macaulay [Mac27] and Stanley [Sta96, Sta77]. With some additional information taken from Björner, Frankl & Stanley [BFS87] we get all parts of the following theorem.

Theorem 1 (Macaulay-Stanley Theorem)**.** *For an integer vector* $\mathrm{h} = (h_0, h_1, \ldots, h_d)$ *the following are equivalent:*
(a) h *is the h-vector of a CM complex on $[n]$ of dimension $d-1$;*
(b) h *is the h-vector of a pure shifted complex on $[n]$ of dimension $d-1$;*
(c) h *is the f-vector of a multicomplex on $\{w_1, \ldots, w_{n-d}\}$;*
(d) h *is the f-vector of a shifted multicomplex on $\{w_1, \ldots, w_{n-d}\}$;*
(e) h *is an M-sequence with $h_1 \leq n - d$.*

Face numbers of SCM complexes: some necessary conditions. The h-triangle of a shifted complex (more generally, a shellable complex) has a combinatorial interpretation that we now recall, the reader may consult [BW96, BW97] for more information. For a shifted complex Δ, reverse lexicographic order of the facets is a shelling order with restriction map $\mathcal{R}(F) = F \setminus \sigma(F)$, where $\sigma(F)$ is the longest segment $\{s, \ldots, n\} \subseteq F$ if $n \in F$ and is empty otherwise. In particular, one obtains that

$$h_{i,j}(\Delta) \;=\; |\{F \in \mathcal{F}(\Delta) \;:\; |F| = i \;\&\; |\sigma(F)| = i - j\}|. \tag{2}$$

Algebraic shifting is an operator on simplicial complexes that associates to a simplicial complex a shifted complex, preserving many interesting invariants of the complex. We refer the reader to the article by Kalai [Kal02] or the book by Herzog & Hibi [HH11] to see the precise definition and properties. It was shown by Duval [Duv96] that a complex is SCM if and only if algebraic shifting preserves its h-triangle (or, equivalently $\widetilde{h}$-triangle). In particular, the set of h-triangles of SCM complexes coincides with the set of h-triangles of shifted complexes.

Putting these facts together we can deduce the following necessary conditions.

Proposition 2 (c.f. [BW96, Theorem 3.6])**.** *If a triangular integer array* $\widetilde{\mathrm{h}}(\Delta) = (\widetilde{h}_{i,j})_{0 \leq j \leq i \leq d}$ *is the $\widetilde{h}$-triangle of a SCM complex, then*
(a) *every row* $\widetilde{\mathrm{h}}^{[i]} := (\widetilde{h}_{i,0}, \widetilde{h}_{i,1}, \ldots, \widetilde{h}_{i,i})$ *is an M-sequence; and*
(b) $\widetilde{h}_{i,j} \geq \sum\limits_{\ell \leq j} \widetilde{h}_{i+1,\ell}.$

These necessary conditions are, however, not sufficient, as the following example shows.

Example 3. The triangular integer array

$$\widetilde{\mathtt{h}} = \begin{matrix} 1 & & & & \\ 1 & 5 & & & \\ 1 & 4 & 7 & & \\ 1 & 3 & 3 & 4 & \\ 1 & 2 & 0 & 0 & 0 \end{matrix}$$

satisfies the conditions in Proposition 2. However, there exists *no* SCM complex with the given array as its $\widetilde{h}$-triangle.

To see this, assume the contrary and let Δ be a shifted complex with $\widetilde{\mathtt{h}}(\Delta) = \widetilde{\mathtt{h}}$. Let X and Y be the pure 3- and 2-skeleton of Δ, respectively. It follows from equation (2) that X is obtained by taking three iterated cones from a disjoint union of 3 points. Now, looking at f-vectors, it is clear that the underlying graph (1-skeleton) of Y is the same as the underlying graph of X, which is a complete 4-partite graph $K_{3,1,1,1}$. Now, if we remove from Y the smallest vertex in the shifted ordering, the underlying graph becomes $K_{3,1,1}$. However, this graph has only 3 missing triangles, whereas Y has 4 homology facets (i.e., facets F with $\mathcal{R}(F) = F$). Thus we get a contradiction.

Remark Criterion (e) in Theorem 1 shows that in order to decide whether h is the h-vector of a Cohen-Macaulay simplicial complex it suffices to check a certain criterion for pairs of entries of h. The answer is yes if and only if the answer is yes for every pair of consecutive entries.

The same is not true for SCM complexes as shown by the necessity of condition (b) in Proposition 2, as well as by Example 3. More than pairwise checks are needed here.

3 A Combinatorial Correspondence

Correspondences between monomials and sets (or, between sets with repetition and sets without) are well-known in combinatorics. We are going to make crucial use of such a correspondence, namely a more precise and elaborated version of the bijection defined in [BFS87], see Remark 7. We call this the BFS correspondence. It is conveniently explained in terms of lattice paths.

By a *lattice path* from $(0,0)$ to (r,a) we mean a path restricted to east (E) and north (N) steps, each connecting two adjacent lattice points. Thus, a lattice path can be seen as a word $L = L_1, L_2, \ldots, L_{r+a}$ on the alphabet $\{N, E\}$ with the letter N appearing exactly a times. For two lattice paths L and L', let $L < L'$ mean that L never goes above L'. The poset consisting of all lattice paths from $(0,0)$ to (r,a) ordered by this partial order will be denoted by $\mathcal{L}_{r,a}$.

The lattice paths in $\mathcal{L}_{r,a}$ can be encoded in two natural ways: either by the position of the north steps, or by the number of north steps in each column. Thus, for $L \in \mathcal{L}_{r,a}$ let us define:

- $\nu(L)$ is the set of positions within L of its north steps, i.e. $\nu(L) := \{i \; : \; L_i = N\}$;

- $\lambda(L)$ is the monomial $\prod_{i=1}^{r} w_i^{\lambda_i(L)}$, where $\lambda_i(L)$ is the number of north steps of L coordinatized as $(i-1, j) \to (i-1, j+1)$, for some j.

Example 4. Let $L = NEENENNEEEN$. Then $\nu(L) = \{1, 4, 6, 7, 11\}$ and $\lambda(L) = w_1 w_3 w_4^2$, see Figure 1.
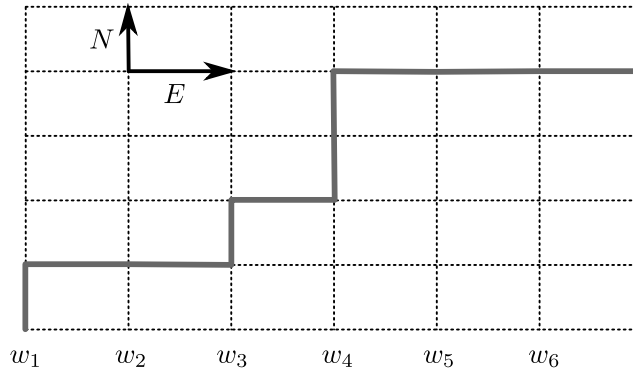

Figure 1: The lattice path $L = NEENENNEEEN$ from $(0,0)$ to $(6,5)$.

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A few more definitions are needed. Recall that an *order ideal* Q in a poset P is a subset $Q \subseteq P$ such that if $x \in Q$ and $z < x$ then $z \in Q$. We use the following notation:

- $\binom{[r+a]}{a}$ = the set of a-element subsets of $\{1, 2, \ldots, r+a\}$
- $\left(\binom{W_r}{\leq a}\right)$ = the set of monomials of degree $\leq a$ in indeterminates $W_r = \{w_1, \ldots, w_r\}$.

  We leave to the the reader to verify the following simple observations.

Proposition 5. *The following holds true:*

(a) *The map ν induces a bijection between shifted set families in $\binom{[r+a]}{a}$ and order ideals in $\mathcal{L}_{r,a}$.*

(b) *The map λ induces a bijection between shifted multicomplexes in $\left(\binom{W_r}{\leq a}\right)$ and order ideals in $\mathcal{L}_{r,a}$.* □

Now, let a be a positive integer and m a monomial on W_r such that $\deg \mathrm{m} \leq a$. Define $\varphi^a(\mathrm{m})$ to be the a-subset $\nu\lambda^{-1}(\mathrm{m})$ of $[r + a]$. We drop the integer a from the notation whenever there is no danger of confusion. Also, let ψ be the inverse of φ. The situation is illustrated in the following diagram of bijective maps:

$$
\begin{array}{ccc}
 & \mathcal{L}_{r,a} & \\
{}^{\lambda}\!\swarrow & & \downarrow{}^{\nu} \\
\left(\binom{W_r}{\leq a}\right) & \begin{array}{c} \xrightarrow{\ \varphi\ } \\ \xleftarrow{\ \psi\ } \end{array} & \binom{[r+a]}{a}
\end{array}
$$

Proposition 6 (BFS correspondence)**.**

(a) *The map $\varphi := \nu\lambda^{-1}$ induces a bijection $\overline{\varphi}$, with inverse $\overline{\psi}$, between shifted multicomplexes in $\left(\binom{W_r}{\leq a}\right)$ and shifted set families in $\binom{[r+a]}{a}$.*

(b) *For a pure shifted $(a-1)$-complex Δ with facets $\mathcal{F}(\Delta)$, one has $\mathrm{h}(\Delta) = \mathrm{f}(\overline{\psi}(\mathcal{F}(\Delta)))$.*

Proof. The first part follows from Proposition 5. For the second part, observe that for a facet F of Δ, the cardinality of its restriction $\mathcal{R}(F)$, as discussed in connection with equation (2), is equal to the number of N steps in the last column of $\nu^{-1}(F)$. Hence,

$$|\mathcal{R}(F)| \ = \ |F| - |\sigma(F)| \ = \ \deg\psi(F).$$

This implies that $\mathrm{h}_i(\Delta) = \mathrm{f}_i(\overline{\psi}(\mathcal{F}(\Delta)))$ for all i. □

Remark 7. Our map φ from monomials to sets can be shown to be identical to the map φ defined in [BFS87, page 30], up to relabeling (reversing the order of vertices and monomials).

It was shown in [BFS87] for multicomplexes M, that M compressed $\Rightarrow \overline{\varphi}(\mathrm{M})$ shellable. Also, we have seen here that M shifted $\Rightarrow \overline{\varphi}(\mathrm{M})$ shifted. Since compressed $\Rightarrow$ shifted $\Rightarrow$ shellable, the second implication strengthens the previous one at both ends of the implication arrow.

Definition 8. Let a be a positive integer and M a shifted multicomplex on W_r of degree less than or equal to a. Define $\Phi^a(\mathrm{M}) = \Phi(\mathrm{M})$ to be the simplicial complex whose set of facets, $\mathcal{F}(\Phi(\mathrm{M}))$, is $\overline{\varphi}(\mathrm{M})$. Also, for a pure shifted $(a-1)$-complex Δ, set $\Psi(\Delta)$ to be the multicomplex consisting of monomials $\psi(F)$, for all facets F of Δ.

Let a be a positive integer and M a shifted multicomplex on W_r of degree less than or equal to a. Define the *a-cone* $\mathscr{C}^a_{r+1}\mathrm{M}$ of M to be

$$\mathscr{C}^a_{r+1}\mathrm{M} \ = \ \left\{w^\ell_{r+1} \cdot \mathrm{m} \ : \ \mathrm{m} \in \mathrm{M} \text{ and } \deg \mathrm{m} + \ell < a\right\}.$$

We will drop the indices $r+1$ and a from the notation, when it is clear from the context. The cone construction on multicomplexes can be seen as a non-square-free analogue of the topological cone. However, it is more useful to see it as an analogue of yet another combinatorial construction; the codimension one skeleton of a simplicial complex.

Proposition 9. *Let* M *be a shifted multicomplex on* W_r *of degree less than or equal to* a. *Then the set* $\mathscr{C}^a_{r+1}$M *is a shifted multicomplex on* W_{r+1}. *Furthermore,* $\Phi^{a-1}(\mathscr{C}^a_{r+1}M)$ *is the* $(a-2)$*-skeleton of* $\Phi^a($M$)$.

Proof. Obviously, $\mathscr{C}^a_{r+1}$M is a pure shifted multicomplex on W_{r+1} of degree $a-1$. Set $\Delta = \Phi^a($M$)$. Then the facets of the codimension one skeleton of Δ are

$$\mathcal{F}(\Delta^{(a-2)}) \;=\; \{F \setminus j \;:\; F \in \mathcal{F}(\Delta) \;\&\; j \in \sigma(F)\}.$$

Let F be a facet of Δ and j an element in $\sigma(F)$. Observe that if $\nu^{-1}(F) = L_1, \ldots, L_{r+a}$, then $L_j = N$ and that $\nu^{-1}(F \setminus j)$ is the lattice path from $(0,0)$ to $(r+1, a-1)$ obtained by changing L_j to an E step. On the level of monomials this is the same as multiplying by a suitable power of w_{r+1}. $\qquad\square$

We wish to extend the BFS bijection to the realm of not-necessarily-pure shifted complexes, the motivation being to make this useful tool available for SCM complexes. To do so we need the following definition.

Definition 10 (Metacomplex). A *d-metacomplex* is a sequence $\mathscr{M} = ($M$^{[0]},$M$^{[1]}, \ldots,$M$^{[d]})$ of multicomplexes on W such that
(a) M$^{[i]}$ is a multicomplex on $\{w_1, \ldots, w_{n-i}\}$ of degree less than or equal to i, for all $0 \leq i \leq d$; and
(b) $\mathscr{C}^i$M$^{[i]} \subseteq$ M$^{[i-1]}$, for all $1 \leq i \leq d$.
Also, define the *f-triangle* of $\mathscr{M}$ to be the triangular integer array $\mathtt{f}(\mathscr{M}) = (f_{i,j})_{0 \leq j \leq i \leq d}$, where $f_{i,j}(\mathscr{M})$ is the number $f_j($M$^{[i]})$ of degree j monomials in M$^{[i]}$. A metacomplex is *shifted* if all underlying multicomplexes M$^{[i]}$ are.

For a d-metacomplex $\mathscr{M}$, let $\overline{\Phi}(\mathscr{M})$ be the union

$$\overline{\Phi}(\mathscr{M}) = \bigcup_{i=0}^{d} \left\{ \varphi^i(\mathrm{m}) \;:\; \mathrm{m} \in \mathrm{M}^{[i]} \right\}$$

of subsets of $[n]$. It follows by Proposition 9 that the shadow of the collection of i-sets in $\overline{\Phi}(\mathscr{M})$ is contained in the collection of $(i-1)$-sets, for all $i \in [d]$. Thus, $\overline{\Phi}(\mathscr{M})$ is a shifted $(d-1)$-complex on $[n]$. Also, Proposition 6(b) implies that the $\widetilde{h}$-triangle of $\overline{\Phi}(\mathscr{M})$ coincides with the f-triangle of $\mathscr{M}$.

Conversely, for a shifted $(d-1)$-complex Δ on $[n]$ the sequence

$$\overline{\Psi}(\Delta) \;:=\; (\Psi(\Delta^{[0]}), \Psi(\Delta^{[1]}), \ldots, \Psi(\Delta^{[d]}))$$

is a metacomplex, whose f-triangle coincides with the $\widetilde{h}$-triangle of Δ. Summarizing, we have established this:

Proposition 11 (Extended BFS correspondence). *The pair* $(\overline{\Phi}, \overline{\Psi})$ *is a bijection between shifted d-metacomplexes on W and shifted $(d-1)$-complexes on $[n]$. Moreover, one has* $\widetilde{\mathtt{h}}(\Delta) = \mathtt{f}(\overline{\Psi}(\Delta))$. $\quad\square$

The extended BFS correspondence can also be derived in terms of lattice paths. Namely, let $\widehat{\mathcal{L}}_{r,a}$ be the set of all $\{N, E\}$ lattice paths beginning at $(0,0)$ and ending at some point among $(n-j, j)$, $0 \leq j \leq d$. Then the order ideals in $\widehat{\mathcal{L}}_{r,a}$ correspond bijectively to shifted d-metacomplexes on W on the one hand and to shifted $(d-1)$-complexes on $[n]$ on the other.

4 Face Numbers of SCM Complexes: A Numerical Characterization

In this section we give a numerical characterization of possible $\widetilde{h}$-triangles of SCM complexes. For that purpose, we need to consider special kinds of integer systems $\mathcal{D} = \{q^{\mathrm{m}}\}_{\mathrm{m}}$, indexed by monomials m of degree less than or equal to t on a set $W_s = \{w_1, w_2, \ldots, w_s\}$ of variables.

Definition 12. Let s and t be integers, $1 \leq s, t \leq d$. An $M_{s,t}$-*array* is a function $q : \left(\binom{W_s}{\leq t} \right) \to \mathbb{Z}^+$ (whose values we write q^{m} rather than the conventional $q(\mathrm{m})$), such that
(1) if $\deg(\mathrm{m}) = t - \ell$ and $\mathrm{m}' = u_j \cdot (\mathrm{m}/u_i)$ for some $i < j$ such that u_i divides m, then $q^{\mathrm{m}} \leq q^{\mathrm{m}'}$;
(2) if $\deg \mathrm{m} = t$, then $q^{\mathrm{m}} = 1$;
(3) if $\mathrm{m}' = \mathrm{m} \cdot u_j$ for some $j \in [s]$ and $\deg(\mathrm{m}) = t - \ell$, then $\partial^\ell(q^{\mathrm{m}}) \leq q^{\mathrm{m}'}$.
Furthermore, let $\mathrm{h} = (h_0, h_1, \ldots, h_d)$ be an M-sequence and $r \in \mathbb{Z}^+$ an integer. An $M_{s,t}(\mathrm{h})$-*composition* of r is an $M_{s,t}$-array $\mathcal{D} = \{q^{\mathrm{m}}\}_{\mathrm{m}}$ such that
(4) $h_\ell \leq q^{\mathrm{m}}$ if $\deg(\mathrm{m}) = t - \ell$;
(5) $\sum_{\mathrm{m}} q^{\mathrm{m}} = r$.
For an $M_{s,t}$-array $\mathcal{D} = \{q^{\mathrm{m}}\}_{\mathrm{m}}$, we let $\Sigma_s \mathcal{D}$ be the sum of all q^{m} such that u_s divides m. That given, we define:
(6) $\rho_{s,t}(r; \mathrm{h}) = \min \{ \Sigma_s \mathcal{D} : \mathcal{D} \text{ is an } M_{s,t}(h)\text{-composition of } r \}$.
(7) An $M_{s,t}(\mathrm{h})$-composition $\mathcal{D}$ of r is said to be a *minimal* if $\Sigma_s \mathcal{D} = \rho_{s,t}(r; \mathrm{h})$.

Example 13. Let $\mathrm{h} = (1, 4, 9, 4, 1)$ and $r = 22$. Then

$$\begin{aligned}
\mathcal{D}_1 &= \{ q^{\mathbf{1}} = 10, q^{u_1} = 4, q^{u_2} = 5, q^{u_1^2} = q^{u_2^2} = q^{u_1 u_2} = 1 \}, \quad \text{and} \\
\mathcal{D}_2 &= \{ q^{\mathbf{1}} = 9, q^{u_1} = 5, q^{u_2} = 5, q^{u_1^2} = q^{u_2^2} = q^{u_1 u_2} = 1 \}
\end{aligned}$$

are two minimal $M_{2,2}(\mathrm{h})$-compositions of 22. Whereas,

$$\mathcal{D}_3 = \{ q^{\mathbf{1}} = 9, q^{u_1} = 4, q^{u_2} = 6, q^{u_1^2} = q^{u_2^2} = q^{u_1 u_2} = 1 \}$$

is a non-minimal $M_{2,2}(\mathrm{h})$-compositions of 22.

Clearly, for an integer r and a triple (h, t, s), as in Definition 12 and such that r is greater than or equal to $\sum_{i=0}^{t} \binom{s+i-1}{i} h_{t-i}$, an $M_{s,t}(\mathrm{h})$-composition of r exists. Hence, the quantity $\rho_{s,t}(r; \mathrm{h})$ is well-defined. However, there is a canonical way to obtain a minimal composition that we now discuss.

Remark 14. Note that for $s = 1$ condition (1) is void and the array is linear. So, by conditions (2) and (3) the concept is then equivalent to that of an ordinary M-sequence.

Let us first fix some notation. For a positive integer p with ℓ-representation

$$p = \binom{a_\ell}{\ell} + \binom{a_{\ell-1}}{\ell-1} + \ldots + \binom{a_e}{e},$$

where $a_\ell > a_{\ell-1} > \ldots > a_e \geq e \geq 1$. Define

$$\partial^{\langle \ell, j \rangle}(p) = \binom{a_\ell - j}{\ell - j} + \binom{a_{\ell-1} - j}{\ell - j - 1} + \ldots + \binom{a_e - j}{e - j}.$$

In particular, $\partial^{\langle \ell, 0 \rangle}(p) = p$ and $\partial^{\langle \ell, 1 \rangle}(p) = \partial^\ell(p)$. Note that $\partial^{\langle \ell, j \rangle}(p)$ is a lower bound on the number of monomials of degree $\ell - j$ in a multicomplex M with $f_\ell(\mathrm{M}) = p$.

Let us define a linear order on the monomials of degree less than or equal to t on the set U_s of variables. For all i, set $1 <_i u_i <_i u_i^2 <_i \ldots <_i u_i^t$. Finally set $<_\pi$ to be the product order of all $<_i$ induced by $u_1 < \ldots < u_s$. Also, for a monomial m of degree $t - \ell$ on U_s and a non-negative integer $j \leq t$ define

$$c_j(\mathrm{m}) = |\{ \text{monomials } \mathrm{m}' \text{ on } U_{s,t-j} : \deg \mathrm{m}' = t - j \ \& \ \mathrm{m} <_\pi \mathrm{m}' \}|.$$

Construction 15. Let r, h, t and s be as in Definition 12. We construct a minimal $M_{s,t}(\mathrm{h})$-composition of r inductively as follows.
(1) Set $q^{\mathbf{1}}$ to be the maximum integer p such that

$$\sum_{j=1}^{t} \binom{s+j-1}{j} \cdot \max\{ h_{t-j}, \partial^{\langle t, j \rangle}(p) \} \leq r - p.$$

(2) Let m be a monomial of a positive degree $t-\ell$ and assume that $q^{m'}$ is defined for all monomials $m' <_\pi m$. Set q^m to be the maximum integer p such that the quantity

$$\sum_{m' <_\pi m} q^{m'} + \sum_{j=0}^{\ell} c_{\ell-j}(m) \cdot \max\{h_{\ell-j}, \partial^{\langle \ell,j \rangle}(p)\} + \sum_{j=\ell+1}^{t} c_j(m) \cdot \max\{q^{m'} : \deg m' = t-j \ \& \ m' <_\pi m\}$$

is not greater than $r - p$.

It is not difficult to see that the construction above yields a minimal $M_{s,t}(h)$- composition of r. This minimal composition will be called *the regular $M_{s,t}(h)$-composition of r*.

The following is our main result.

Theorem 16. *A triangular integer array* $\widetilde{h} = (\widetilde{h}_{i,j})_{0 \leq j \leq i \leq d}$ *is the $\widetilde{h}$-triangle of a sequentially CM complex if and only if*
(a) *Every row* $h^{[i]} = (\widetilde{h}_{i,0}, \widetilde{h}_{i,1}, \ldots, \widetilde{h}_{i,i})$ *is an M-sequence;*
(b) $\widetilde{h}_{i,j} \geq \sum_{\ell \leq j} \widetilde{h}_{i+1,\ell}$;
(c) $\rho_{j,d-i}(\widetilde{h}_{i,j}; h^{[d]}) \leq \widetilde{h}_{i,j-1}$.

Proof of the necessity part of Theorem 16. The conditions (a) and (b) are already discussed in Proposition 2. We shall prove the necessity of condition (c).

Let Δ be a shifted $(d-1)$-complex and $\mathscr{M} := (M^{[0]}, M^{[1]}, \ldots, M^{[d]})$ its associated metacomplex on W. Let us also denote by $Q_{i,j}$ the set of all monomials in $M^{[i]}$ of degree j. In particular, the cardinality of the set $Q_{i,j}$ is equal to $\widetilde{h}_{i,j}$. Now, for a monomial m on $U = \{w_{n-d+1}, \ldots, w_{n-i}\}$, consider the set

$$Q_{i,j}^m = \{p = p(w_1, \ldots, w_{n-i}) \in Q_{i,j} : p(1, \ldots, 1, w_{n-d}, \ldots, w_{n-i}) = m\}.$$

We denote by $q_{i,j}^m$ the cardinality of the set $Q_{i,j}^m$.

Claim. *Setting $u_t := w_{n-d+t}$, for all $t \in [d-i]$, the system $\mathcal{D} = \{q_{i,j}^m\}_m$ is an $M_{j,d-i}(h^{[d]})$-composition of $\widetilde{h}_{i,j}$.*

Proof of the claim. First observe that the sets $Q_{i,j}^m$ form a partition of $Q_{i,j}$. Hence, the condition (5) of Definition 12 is satisfied. Now, let m be a monomial of degree $j - \ell$ on U and $m' = u_k(m/u_r)$ for some r and k such that $r < k \leq d - i$ and u_r divides m. It follows from Proposition 11 and Definition 10 that $\widetilde{h}_{d,\ell} \leq q_{i,j}^m$. Also, since $M^{[i]}$ is shifted for every $p \in Q_{i,j}^m$ one has $u_k \cdot u_r^{-1} \cdot p \in Q_{i,j}^{m'}$. Thus, we have $q_{i,j}^m \leq q_{i,j}^{m'}$ and the condition (1) is also valid.
Finally, set $m' = u_k \cdot m$ for some $k \leq d - i$. Let p be a monomial in $Q_{i,j}^m$. For every w in $\{w_1, \ldots, w_{n-d}\}$ that divides p, the monomial $u_k \cdot (p/w)$ is in $Q_{i,j}^{m'}$, since $M^{[i]}$ is shifted. Hence, the shadow of the collection $\{p/m : p \in Q_{i,j}^m\}$ of monomials is contained in $\{p'/m' : p' \in Q_{i,j}^{m'}\}$. This verifies the condition (3) of Definition 12. Therefore, $\mathcal{D} = \{q_{i,j}^m\}_m$ is a $M_{j,d-i}(h^{[d]})$-composition of $\widetilde{h}_{i,j}$. $\square$

To complete the proof of necessity, for every monomial m on U that is divisible by w_{n-i}, set $m' = m/w_{n-i}$. The division map

$$\times w_{n-i}^{-1} : Q_{i,j}^m \to Q_{i,j-1}^{m'}$$

is an injection, since $M^{[i]}$ is a multicomplex. Hence, we have

$$\Sigma_{d-i} \mathcal{D} = \sum_{w_{n-i}|m} q_{i,j}^m \leq \sum_{m'} q_{i,j-1}^{m'} = \widetilde{h}_{i,j-1}.$$

Therefore we have $\rho_{j,d-i}(\widetilde{h}_{i,j}; h^{[d]}) \leq \widetilde{h}_{i,j-1}$, as desired. $\square$

Proof of the sufficiency part of Theorem 16. Let $\widetilde{\mathbf{h}}$ be a triangular integer array satisfying Conditions (a), (b) and (c) of the statement. In the light of Proposition 11, it suffices to construct a metacomplex $\mathscr{M}$ such that $\mathtt{f}(\mathscr{M}) = \widetilde{\mathbf{h}}$. We construct $\mathscr{M}$ as follows:

(1) Let $\mathrm{M}^{[d]}$ be the compressed multicomplex on $\{w_1, w_2, \ldots, w_{n-d}\}$ consisting of the first $\widetilde{h}_{d,j}$ monomials of degree j in the reverse lexicographic order, for all $0 \leq j \leq d$.

(2) Let i be an integer less than d. We shall construct $\mathrm{M}^{[i]}$. For $j \leq i$, let $\mathcal{D}_{i,j} = \{q_{i,j}^{\mathrm{m}}\}_{\mathrm{m}}$ be the regular $M_{j,d-i}(\mathrm{h}^{[d]})$-composition of $\widetilde{h}_{i,j}$. Consider the change of variables $u_t \to w_{n-d+t}$, for $t \in [d-i]$. We will use the same notation m to denote the image of m under this change of variables, this should not lead to any confusion.

Now, for a monomial m of degree ℓ on $\{w_{n-d+1}, \ldots, w_{n-i}\}$, let $P_{i,j}^{\mathrm{m}}$ be the set of the first $q_{i,j}^{\mathrm{m}}$ monomials of degree $j - \ell$ on $\{w_1, \ldots, w_{n-d}\}$ in reverse lexicographic order. Also, let

$$Q_{i,j} \;=\; \bigcup_{\mathrm{m}} \left\{ \mathrm{m} \cdot \mathrm{p} \; : \; \mathrm{p} \in P_{i,j}^{\mathrm{m}} \right\}.$$

Finally, we set $\mathrm{M}^{[i]} = \bigcup_{j=0}^{i} Q_{i,j}$.

Clearly, the number of elements of degree j in $\mathrm{M}^{[i]}$ is $\widetilde{h}_{i,j}$. Also, given that the $\mathrm{M}^{[i]}$'s are shifted multicomplexes, it follows from Condition (b) that $\mathscr{C}^{i+1}\mathrm{M}^{[i+1]} \subseteq \mathrm{M}^{[i]}$. Thus, it only remains to show that $\mathrm{M}^{[i]}$ is a shifted multicomplex. We first show that $Q_{i,j}$ is a shifted family of monomials, for all j. Let p be a monomial in $Q_{i,j}^{\mathrm{m}}$, w_r and w_k two variables in $\{w_1, \ldots, w_{n-i}\}$ such that $k < r$ and w_k divides p. We shall show that $w_r \cdot (\mathrm{p}/w_k) \in Q_{i,j}$. Consider the following cases:

Case 1. $(k < r \leq n - d)$ If $\mathrm{p}' = \mathrm{p}/\mathrm{m}$, then we have $w_r \cdot (\mathrm{p}'/w_k) \in P_{i,j}^{\mathrm{m}}$, since $P_{i,j}^{\mathrm{m}}$ is shifted. Hence, $w_r \cdot (\mathrm{p}/w_k) \in Q_{i,j}^{\mathrm{m}}$.

Case 2. $(k \leq n - d < r \leq n - i)$ Note that the shadow of $P_{i,j}^{\mathrm{m}}$ is contained in $P_{i,j}^{\mathrm{m}'}$ by Condition 12(3), where $\mathrm{m}' = w_r \cdot \mathrm{m}$. Thus, $w_r \cdot (\mathrm{p}/w_k) \in Q_{i,j}^{\mathrm{m}'}$.

Case 3. $(k \leq n - d < r \leq n - i)$. Condition (1) of Definition 12 implies that $P_{i,j}^{\mathrm{m}}$ is contained in $P_{i,j}^{\mathrm{m}'}$, for $\mathrm{m}' = w_r \cdot (\mathrm{m}/w_k)$. In particular, $w_r \cdot (\mathrm{p}/w_k) \in Q_{i,j}^{\mathrm{m}'}$ and $Q_{i,j}$ is a shifted family.

Finally, assume that p is a monomial in $Q_{i,j}$ and w is a variable dividing p. The shifted property insures that $w_{n-i} \cdot (\mathrm{p}/w) \in Q_{i,j}$. However, it follows from Condition (c) that $\mathrm{p}/w \in Q_{i,j-1}$. This shows that $\mathrm{M}^{[i]}$ is a multicomplex. $\qquad\square$

5 Betti Tables of Componentwise Linear Ideals

In this section we obtain a characterization of the possible Betti diagrams of componentwise linear ideals in a polynomial ring over a field of arbitrary characteristic. We start by recalling some definitions and refer the reader to the book by Herzog & Hibi [HH11] for undefined terminology.

An ideal I is said to have an *r-linear resolution* if $b_{s,s+\ell}(I) = 0$ for all $\ell \neq r$. For an ideal I, let $I_{(r)}$ be the ideal generated by all monomials of degree r in I. Then I is called *componentwise linear* if $I_{(r)}$ has an r-linear resolution for all r.

For square–free monomial ideals the notion of componentwise linearity is dual to sequential Cohen–Macaulayness in the sense that: the Stanley–Reisner ideal I_Δ of a complex Δ is componentwise linear if and only if its Alexander dual Δ^* is SCM. In particular, the Stanley–Reisner ideal of a shifted complex is componentwise linear; such an ideal is called *square–free strongly stable*. The square–free strongly stable ideals are square–free analogues of strongly stable ideals. Recall that, a monomial ideal $I \subseteq S$ is said to *strongly stable* if for every monomial u in the minimal set $\mathcal{G}(I)$ of monomial generators of I and all $i < j$ such that x_j divides u, one has $x_i \cdot (u/x_j)$ is in I.

Observation 17 (Herzog, Sharifan & Varbaro, [HSV14])**.** The set of Betti tables of componentwise linear ideals in a polynomial ring over a field of an arbitrary characteristic coincides with those of the strongly stable ideals.

In characteristic zero, it is known [HH11, Theorem 8.2.22] that componentwise linearity can be characterized as ideals with stable Betti table under (reverse lexicographic) generic initial ideal. The interesting part of the observation is that the characterization of the Betti tables does not depend on the characteristic. We do not rewrite the observation here, instead we refer the reader to [HSV14, page 1879] for more details.

Proposition 18. *The set of all Betti tables of r-regular componentwise linear ideals in the polynomial ring $\Bbbk[x_1, \ldots, x_n]$ coincides with the set of all Betti tables of r-regular square–free strongly stable ideals in the polynomial ring $\Bbbk[x_1, \ldots, x_{n+r-1}]$.*

Proof. Note that, the Betti table of an ideal depends only on the set of generators, in the sense that if $I \subseteq \Bbbk[x_1, \ldots, x_n]$ and J is the ideal generated by the set $\mathcal{G}(I)$ of generators of I in the polynomial ring $\Bbbk[x_1, \ldots, x_{n+r}]$, then I and J have the same Betti tables. Now the conclusion follows from [HH11, Lemma 11.2.5] and [HH11, Lemma 11.2.6]. $\square$

Let I be a square–free strongly stable ideal in S. For $\mathrm{u} \in \mathcal{G}(I)$, let us denote by $m(\mathrm{u})$ the biggest index t such that x_t divides u. If $d = \{\min \deg \mathrm{u} \ : \ \mathrm{u} \in \mathcal{G}(I)\}$, then for $\ell \geq d$ define:

$$m_{k,\ell}(I) \ = \ |\{\mathrm{u} \in \mathcal{G}(I) \ : \ \deg \mathrm{u} = \ell \ \& \ m(\mathrm{u}) = k + \ell - 1\}|.$$

Clearly, $m_{k,\ell} = 0$ if $k + \ell$ is greater than $n + 1$. Thus we may think of the collection of doubly indexed m-numbers as a triangular array. The triangular integer array $\mathtt{m}(I) =: (m_{k,\ell})$; $1 \leq k \leq n - \ell + 1 \leq n - d + 1$, is called the *reduced array of generators* of I.

The square–free version of Eliahou–Kervaire implies (see [HH11, Subsection 7.2]) that

$$b_{s,s+\ell}(I) \ = \ \sum_{k=s-1}^{n} \binom{k}{s} m_{k,\ell}(I),$$

or equivalently

$$\sum_{s \geq 0} b_{s,s+\ell}(I) t^s \ = \ \sum_{s \geq 0} m_{s+1,\ell}(I)(1+t)^s. \tag{3}$$

In particular, the characterisation of the possible Betti tables of square–free strongly stable ideals is equivalent to characterizing the possible reduced arrays of generators. Following [HSV14] for a square–free strongly stable ideal I we also consider doubly indexed μ-numbers defined recursively by the following relation

$$m_{\ell,k} \ = \ \mu_{\ell,k} - \sum_{q=1}^{\ell} \mu_{q,k-1}. \tag{4}$$

The triangular integer array $\widetilde{\mathtt{m}}(I) = (\mu_{\ell,k})$; $1 \leq k \leq n - \ell + 1 \leq n - d + 1$, is called the *array of generators* of I.

The task of characterizing all possible (reduced) arrays of generators of square–free strongly stable ideals, however, translates nicely into combinatorics as follows.

Lemma 19. *Let Δ be a shifted simplicial complex on $[n]$. Then*

$$m_{s+1,k}(I_\Delta) \ = \ h_{n-k,s}(\Delta^*).$$

In particular, the array of generators of I_Δ is the same as $\widetilde{h}$-triangle of Δ^* (up to a suitable rotation).

Proof. For a facet F in a shifted simplicial complex on $[n]$, let ℓ_F be the smallest integer such that $\ell_F \in \sigma(F)$, if $\sigma(F)$ is non-empty and otherwise set $\ell_F = n + 1$. It follows from equation (2) that

$$h_{n-k,s}(\Delta^*) \ = \ |\{F \in \mathcal{F}(\Delta^*) \ : \ |F| = n - k \ \& \ \ell_F = s + k + 1\}|.$$

Now, observe that the complement map $F \mapsto F^c$ induces a bijection between $\mathcal{F}(\Delta^*)$ and $\mathcal{G}(I_\Delta)$ with the property that: if u is the image of F, then $\deg \mathrm{u} + |F| = n$ and $\ell_F - 1 = m(\mathrm{u})$. Hence, we obtain

$$h_{n-k,s}(\Delta^*) \ = \ |\{\mathrm{u} \in \mathcal{G}(I_\Delta) \ : \ \deg \mathrm{u} = k \ \& \ m(\mathrm{u}) = s + k\}| = m_{s+1,k}(I_\Delta).$$

The last part of the statement follows by comparing equations (1) and (4). $\square$

The following corollary first appeared in [HRW99, Proposition 12]. Unfortunately, there is a misprint in the statement in the published version of the paper.

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Corollary 20. *Let Δ be sequentially Cohen-Macaulay. Then*

$$\sum_{i\geq 0} b_{i,i+j}(I_{\Delta^*})t^i \ = \ \sum_{i\geq 0} h_{n-j-1,i}(\Delta)(1+t)^i.$$

Proof. Using algebraic shifting, it is enough to prove the result for the special case of shifted complexes. However, in this case the result follows from equation (3) and Lemma 19. $\qquad\square$

Theorem 21. *A triangular integer array $\widetilde{\mu} = (\mu_{\ell,k})$; $1 \leq k \leq n - \ell + r \leq n - d + r$, is the array of generators of an r-regular componentwise linear ideal with minimum degree of a generator equals to d on S if and only if*
(a) *Every column $\mu^{[j]} = (\widetilde{\mu}_{1,j}, \widetilde{\mu}_{2,j}, \ldots, \widetilde{\mu}_{n+r-j,j})$ is an M-sequence;*
(b) *$\widetilde{\mu}_{i,j} \geq \sum_{\ell \leq i} \widetilde{\mu}_{\ell,j-1}$;*
(c) *$\rho_{i,j-d+1}(\widetilde{\mu}_{i+1,j}; \mu^{[d]}) \leq \widetilde{\mu}_{i,j}$.*

Proof. This follows from Lemma 19 and Theorem 16. $\qquad\square$

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